Mathematics: analysis and approaches
Standard level
Paper 2

Name

worked solutions

Date: ____________________

1 hour 30 minutes

Instructions to candidates

- Write your name in the box above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your name on each answer sheet and attach them to this examination paper.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.
- A clean copy of the **Mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[80 marks]**.

14 pages

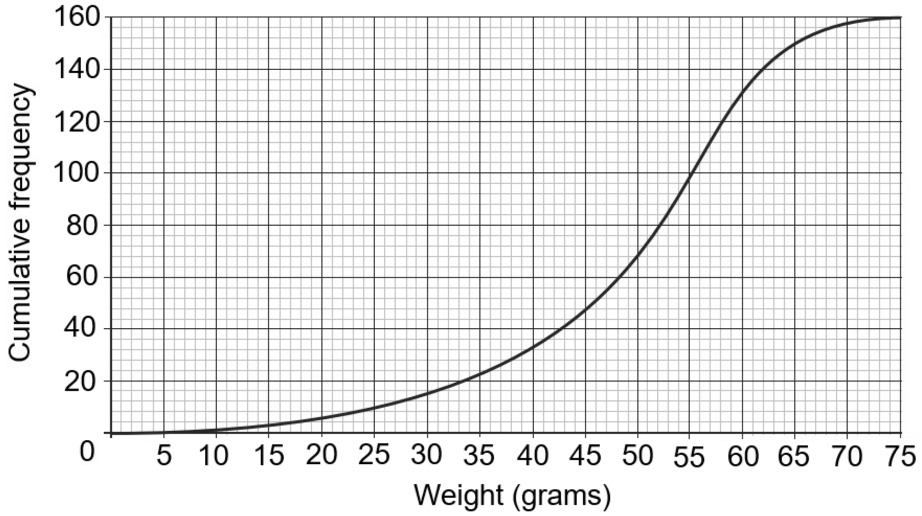
Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, for example if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A (38 marks)

Answer **all** questions in the boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 8]

The cumulative frequency graph below represents the weight in grams of 160 limes picked from a mature Mexican Key lime tree.



(a) Estimate the

(i) median weight of the limes;

(ii) 40^{th} percentile of the weight of the limes. [3]

(b) Estimate the number of limes which weigh more than 50 grams. [2]

(c) The middle 50% of limes weigh between a grams and b grams, where $a < b$. Find the value of a and the value of b. [3]

(a) (i) median weight = 52 grams [accept 52.0 to 52.5]

(ii) $0.4(160) = 64 \Rightarrow$ 40th percentile = 49 grams

(b) 68 limes weigh less than 50 grams; $160 - 68 = 92$

92 limes weigh more than 50 grams

(c) 50% of $160 = 80 \Rightarrow$ middle 50% from 40 to 120

$a = 43$ grams [accept 42.0 to 43.0] $b = 58$ grams [accept 58.0 to 58.5]

2. [Maximum mark: 6]

On the first day of 2024, Noah deposited h dollars in a bank account that earns a nominal annual interest rate of 4.8% compounded **monthly**. Interest is added to the account on the first day of each month. Interest is added for the first time on 1st February 2024.

The amount of money in Noah's account on **the first day of each year** follows a geometric sequence with the common ratio r.

(a) Find the value of r. [3]

(b) If Noah makes no further deposits into or withdrawals from the account, find the **year** in which his account has at least $2h$ dollars in it (double the initial deposit) for the first time. [3]

(a) on 1st day of 2025 : $FV = PV\left(1+\frac{0.048}{12}\right)^{12} = h(1.04907\ldots)$

$r \approx 1.049$ (4 significant figures)

(b) $2h = h\left(1+\frac{0.048}{12}\right)^{12t}$ $t = \#$ of years

$2 = 1.004^{12t} \Rightarrow t \approx 14.469$

hence, the account will have 2h dollars in it in 15 years

2h dollars first in account in year 2039

OR

$PV = 1$

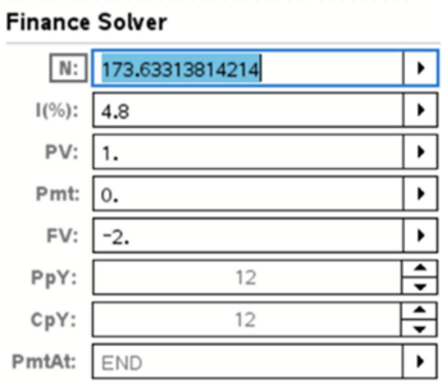
$FV = -2$

$I\% = 4.8$

$P/Y = 4$

$C/Y = 4$

$n = 173.633\ldots$ (payment periods)



$\frac{173.633\ldots}{12} \approx 14.469$ years

2h dollars first in account in year 2039

3. [Maximum mark: 5]

Prove the following statement:

If m and n are both odd integers, then $m+n$ is an even integer.

Let $m = 2a + 1$ and $n = 2b + 1$
where a and b are integers

$m + n = 2a + 1 + 2b + 1$
$= 2(a + b + 1)$

Since a, b and 1 are integers then
the sum $a + b + 1 = k$ must be an integer

hence, $m + n = 2k$
thus, $m + n$ must be an even integer
since it has a factor of 2 (divisible by 2)

4. [Maximum mark: 6]

Consider the diagram below with the measure of the angle at vertex A and lengths of line segments $[\mathrm{AB}]$, $[\mathrm{BC}]$ and $[\mathrm{BD}]$ indicated. Find the length of line segment $[\mathrm{CD}]$.

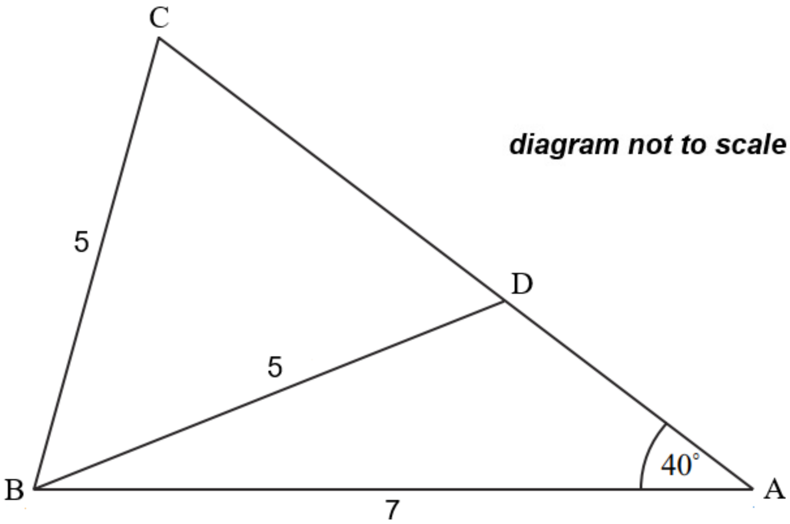
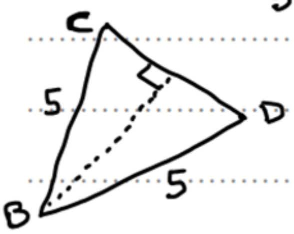


diagram not to scale

Method 1: $\dfrac{\sin C}{7} = \dfrac{\sin 40°}{5}$ sine rule

$\sin C = \dfrac{7\sin 40°}{5} \Rightarrow C = \sin^{-1}\left(\dfrac{7\sin 40°}{5}\right) \approx 64.14...°$



$\cos C = \dfrac{\frac{1}{2}CD}{5} \Rightarrow CD = 2\left[5\cos(64.14...°)\right]$

$CD \approx 4.36$

Method 2: let $AC = x$

$5^2 = 7^2 + x^2 - (2)(7)(x)\cos 40°$ cosine rule

$x^2 - (14\cos 40°)x + 24 = 0 \Rightarrow x^2 - (10.72...)x + 24 = 0$

$x = \dfrac{10.72... \pm \sqrt{(10.72...)^2 - 4(24)}}{2} \Rightarrow x \approx 7.543...$ or $x \approx 3.182...$

let $AD = y$: $5^2 = 7^2 + y^2 - (2)(7)(y)\cos 40°$ [same equation as above]

$y \approx 7.543...$ or $y \approx 3.182...$

hence, $AC \approx 7.543...$ and $AD \approx 3.182...$

$CD = AC - AD = 7.543... - 3.182... \Rightarrow CD \approx 4.36$

5. [Maximum mark: 7]

A particle moves in a straight line. At time t seconds, the particle's displacement from a fixed point O is s meters. The particle's velocity v (in meters per second) is given by $v = \sin\left(\frac{t}{2}\right), t \geq 0$.

When $t = 0$, $s = 2$ meters.

(a) Express the displacement s as a function of time t. [5]

(b) Find the total distance the particle travels from $t = 0$ seconds to $t = 10$ seconds. [2]

(a) $s(t) = \int v(t)\,dt = \int \sin\left(\frac{t}{2}\right) dt = -2\cos\left(\frac{t}{2}\right) + C$

given $s(0) = 2 \Rightarrow -2\cos(0) + C = 2 \Rightarrow C = 4$

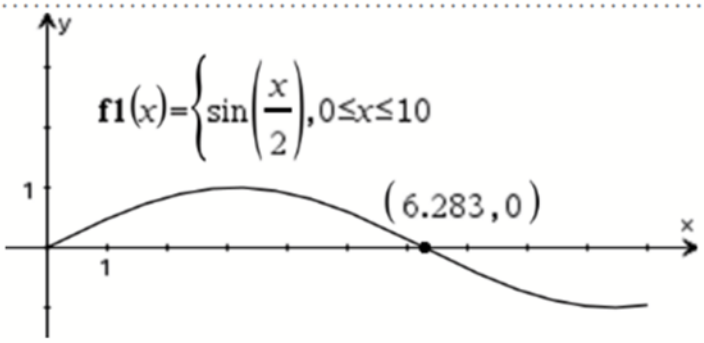
thus, $s(t) = -2\cos\left(\frac{t}{2}\right) + 4$

(b) total distance travelled from $t=0$ to $t=10$ $= \int_0^{10} \left|\sin\left(\frac{t}{2}\right)\right| dt \approx 6.57$ m

OR

total distance $= \int_0^{2\pi} \sin\left(\frac{t}{2}\right) dt - \int_{2\pi}^{10} \sin\left(\frac{t}{2}\right) dt$

$= 4 - (-2.567...) \approx 6.57$ m



6. [Maximum mark: 6]

When $\left(1+\frac{2x}{3}\right)^n$, $n \in \mathbb{N}$, is expanded in ascending powers of x, the coefficient of x^2 is 68.

(a) Find the value of n. [5]

(b) Hence, find the coefficient of x^3. [1]

(a) $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$

coefficient of x^2 for $\left(1+\frac{2x}{3}\right)^n$ is $\binom{n}{2}(1)^{n-2}\left(\frac{2}{3}\right)^2 = 68$

$\frac{n!}{2!\,(n-2)!} \cdot 1 \cdot \frac{4}{9} = 68 \Rightarrow n(n-1) = 306$

$n^2 - n - 306 = 0 \Rightarrow \cancel{n=-17}$ or $n = 18$

(b) coefficient of $x^3 = \binom{18}{3}\left(\frac{2}{3}\right)^3 = 816\left(\frac{8}{27}\right) = \frac{2176}{9}$

$\left[\text{or } 241.\overline{7}\right]$

Do **not** write solutions on this page.

Section B (42 marks)

Answer **all** the questions on the answer sheets provided. Please start each question on a new page.

* worked solution on next page →

7. [Maximum mark: 16]

A large farm produces eggs that are packaged in boxes with 12 eggs in each box. The probability that a single egg is cracked is 0.017. A random box is selected and the 12 eggs in it are inspected.

(a) Find the probability that exactly one egg in the box is cracked. [3]

(b) A box fails inspection if at least two eggs in it are cracked. Find the probability that the randomly selected box passes (does not fail) inspection. [3]

The weights of individual eggs are normally distributed with a mean of 58 grams and a standard deviation of 4.7 grams. Before the eggs are packed in boxes a very large number of them are placed in a sorting bin.

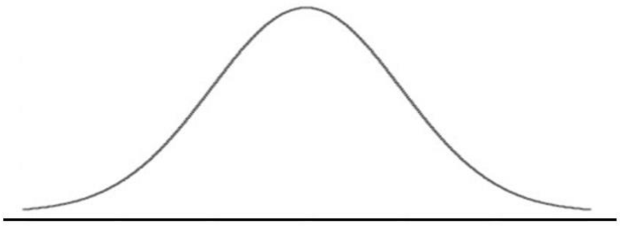
(c) One egg is chosen at random from the bin. Find the probability that this egg

(i) weighs less than 64 grams;

(ii) weighs between 52 grams and 64 grams. [4]

(d) 10% of the eggs in the bin weigh less than w grams.

(i) Copy and complete the following normal distribution diagram, to represent this information, by indicating w, and shading the appropriate region.



(ii) Find the value of w. [4]

(e) An egg selected from the sorting bin is accepted to be put into a box if its weight lies between 52 grams and 64 grams. 12 eggs are randomly selected from the sorting bin. Find the probability that all the eggs are accepted to be put in a box. [2]

[see GDC images for Q.7 solution on next page]

7.

(a) X is random variable representing # of cracked eggs in a box

$$X \sim B(12, 0.017)$$

$$P(x=1) = {}_{12}C_1 (0.017)^1 (1-0.017)^{11}$$

$$\underline{\underline{P(x=1) \approx 0.169}}$$

(b) probability a box passes inspection $= P(X \leq 1) = P(x=0) + P(x=1)$

$$P(x=0) = (1-0.017)^{12} = (0.983)^{12} \approx 0.814033\ldots$$

probability box passes $= 0.814033\ldots + 0.168935\ldots \approx 0.982968\ldots$

$$\approx \underline{\underline{0.983}}$$

(c) X is random variable representing weights of eggs

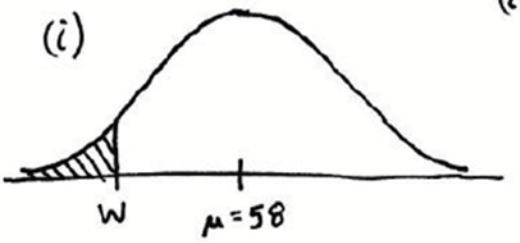
$$X \sim N(58, 4.7^2) \qquad \mu = 58, \ \sigma = 4.7$$

(i) $P(x < 64) \approx \underline{\underline{0.899}}$

(ii) $P(52 < x < 64) \approx \underline{\underline{0.798}}$

(d) $P(x < w) = 0.10$

(i)



(ii) z-value for $P(x < w) = 0.1$ is $z \approx -1.28155$

$$z = \frac{x-\mu}{\sigma} \rightarrow -1.28155 = \frac{x-58}{4.7} \rightarrow x \approx 51.9767$$

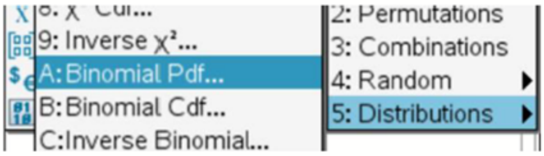
$$\underline{\underline{w \approx 52.0}}$$

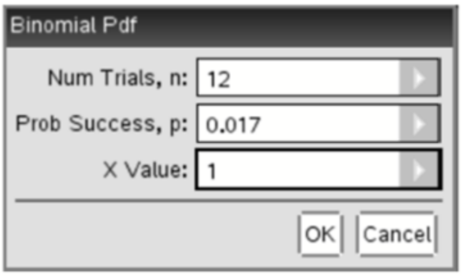
(e) $P(52 < x < 64) \approx 0.798255\ldots$

probability all 12 eggs are "accepted" (weight between 52 and 64 grams) is equal to $(0.798255\ldots)^{12} \approx \underline{\underline{0.0669}}$

Q.7 GDC images

(a)

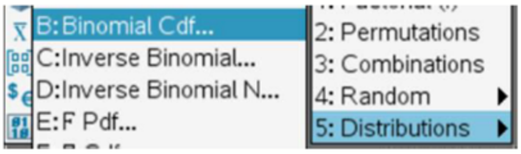


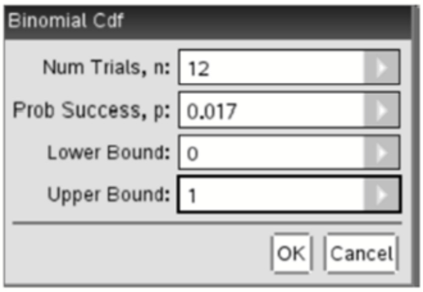


binomPdf(12,0.017,1)   0.168935

nCr(12,1)· 0.017· (1−0.017)^{11} 0.168935

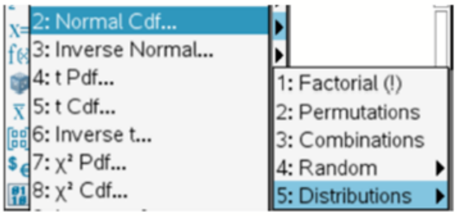
(b)

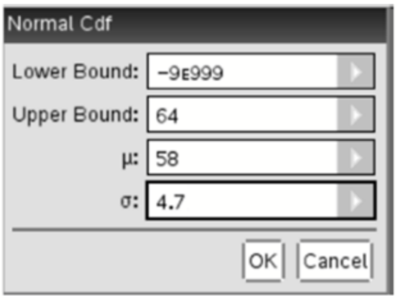




binomCdf(12,0.017,0,1)   0.982968

(c)

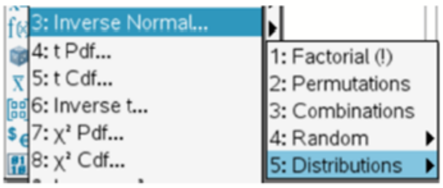




normCdf(−9.E999,64,58,4.7)   0.899127

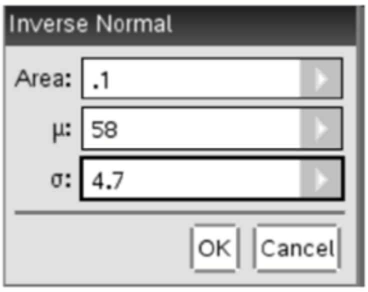
normCdf(52,64,58,4.7)   0.798255

(d) (ii)



invNorm(0.1,0,1)   -1.28155

OR



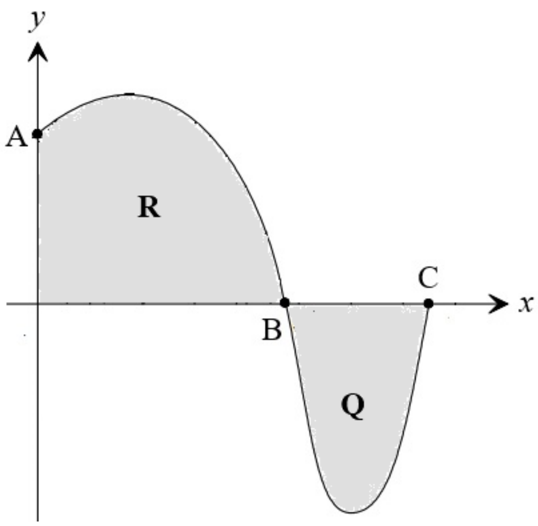
invNorm(0.1,58,4.7)   51.9767

Do **not** write solutions on this page.

* worked solution on next page →

8. [Maximum mark: 12]

The figure below shows the graph of the function $h(x) = \sin(e^x)$ where x is in **radians**. It intersects the y-axis at A, and intersects the x-axis at B and C. The domain of h is $0 \le x \le k$ where k is the x-coordinate of C. Region **R** is bounded by the y-axis, the x-axis and the graph of h. Region **Q** is bounded by the x-axis and the graph of h.



(a) Find the coordinates of A. [2]

(b) The coordinates of B and C can be written in the form $(\ln p, 0)$.

(i) Find the exact value of p for B.

(ii) Find the exact value of p for C.

(iii) Hence, write down the domain of function h. [5]

(c) Write down the range of function h. [1]

(d) (i) Write down the integral which represents the area of region **Q**.

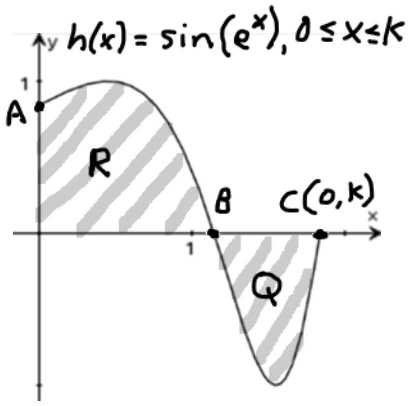
(ii) Find the total area of the shaded regions **R** and **Q**. [4]

8. (a) $x = 0$ at $A \Rightarrow y = \sin(e^0) = \sin(1) \approx 0.84147\ldots$

coordinates of A are $(0, 0.841)$

(b)(i) B: $\sin(e^x) = 0 \Rightarrow e^x = \pi \Rightarrow x = \ln\pi$

$B(\ln p, 0) \Rightarrow p = \pi$



(ii) C: $\sin(e^x) = 0 \Rightarrow e^x = 2\pi \Rightarrow x = \ln 2\pi$

$C(\ln p, 0) \Rightarrow p = 2\pi$

(iii) domain of h: $0 \le x \le \ln 2\pi$

(c) range of $y = \sin x$ is $-1 \le y \le 1$

thus, range of $y = h(x) = \sin(e^x)$ is $-1 \le y \le 1$

(d) (i) area of $Q = \left| \int_{\ln\pi}^{\ln 2\pi} \sin(e^x)\,dx \right|$

(ii) total area of $R + Q = \int_0^{\ln\pi} \sin(e^x)\,dx + \left| \int_{\ln\pi}^{\ln 2\pi} \sin(e^x)\,dx \right|$

≈ 1.34 units^2

Do **not** write solutions on this page.

* worked solution on next page →

9. [Maximum mark: 14]

Consider the rational function $f\left(x\right)=\dfrac{3x-3}{x-6},\ x\neq 6$.

(a) Sketch the graph of f clearly indicating any x-intercept(s), y-intercept(s) and asymptotes, and write down their coordinates and equations. [5]

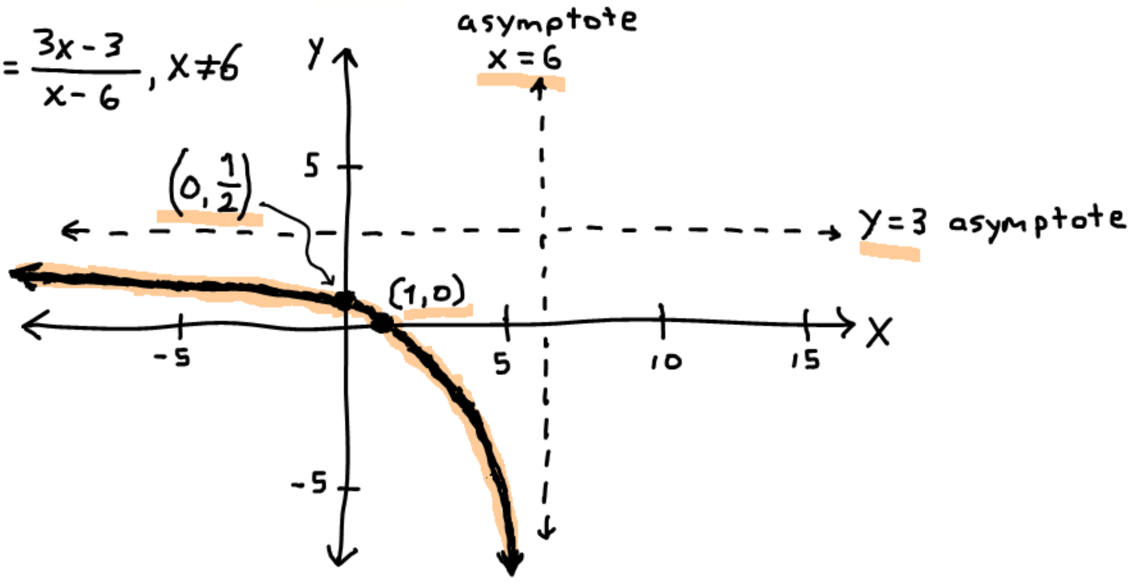
(b) Find an expression for the inverse function $f^{-1}\left(x\right)$. [4]

Consider another rational function $g\left(x\right)=\dfrac{k}{1-x},\ x\neq 1$ where $k\in\mathbb{R}$.

(c) Write down the equation of the horizontal asymptote for the graph of g. [1]

(d) Find the value of k such that $\left(f\circ g\right)\left(x\right)=\dfrac{x+5}{2x},\ x\neq 0$ [4]

9. (a) $f(x) = \dfrac{3x-3}{x-6}, x \neq 6$



(b) $y = \dfrac{3x-3}{x-6}$ switch domain and range $\Rightarrow x = \dfrac{3y-3}{y-6}$ solve for y

$xy - 6x = 3y - 3 \Rightarrow xy - 3y = 6x - 3 \Rightarrow y(x-3) = 6x - 3$

$y = \dfrac{6x-3}{x-3} \Rightarrow f^{-1}(x) = \dfrac{6x-3}{x-3}, x \neq 3$

(c) as $x \to \pm\infty$, $g(x) = \dfrac{k}{1-x} \to 0$

hence, horizontal asymptote is x-axis; equation $y = 0$

(d) $(f \circ g)(x) = \dfrac{3\left(\frac{k}{1-x}\right) - 3}{\frac{k}{1-x} - 6} = \dfrac{x+5}{2x}$

$$\text{LHS} = \frac{\frac{3k}{1-x} - \frac{3(1-x)}{1-x}}{\frac{k}{1-x} - \frac{6(1-x)}{1-x}} = \frac{3k - 3 + 3x}{k - 6 + 6x} = \frac{3(k-1+x)}{k-6+6x}$$

$$\frac{3(k-1+x)}{k-6+6x} = \frac{x+5}{2x} \Rightarrow \frac{k-1+x}{k-6+6x} = \frac{x+5}{6x}$$

equating numerators: $k - 1 + x = x + 5 \Rightarrow k - 1 = 5 \Rightarrow k = 6$

or equating denominators: $k - 6 + 6x = 6x \Rightarrow k - 6 = 0 \Rightarrow k = 6$